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ON TWO-DIMENSIONAL FLUID MOTION THROUGH SPOUTS COMPOSED OF TWO PLANE WALLS.

BY R. A. HARRIS.

IT is well known that the functions sin z, cos z, sinh z, cosh z, transform lines parallel to the axes of x and y, where $z = x + iy$, into systems of confocal ellipses and hyperbolas; also, that if we take the hyperbolas as stream lines, the two-dimensional motion represented by these systems is that of a boundless liquid sheet of uniform thickness flowing through an aperture two units wide made in a thin plane partition or wall extending to infinity in either direction. Helmholtz* discovered the curve systems for the case in which infinite plane walls extend from the edges of the aperture or orifice, and perpendicular to its plane, to infinity in one and the same direction, thus forming the parallel sides of a spout. This paper is concerned with a transformation, equation (1) below, which comprises these two cases together with all intermediate ones.

Assume the equation

$$Z = X + iY = \frac{\epsilon^{\epsilon}(1-\epsilon)^{1-\epsilon}}{\sin \epsilon\pi}\left[e^{(1-\epsilon)z} - e^{-\epsilon z}\right], \tag{1}$$

where ϵ is any real positive quantity ranging from $\frac{1}{2}$ to 0, and $z = x + iy$. The motion is supposed to take place in the Z-plane. The constant factor is, as will soon appear, such that the width of the orifice shall be 2 units in all cases. From equation (1)

$$X = \left(\frac{\epsilon}{1-\epsilon}\right)^{\epsilon}\frac{1-\epsilon}{\sin \epsilon\pi}\left[e^{(1-\epsilon)x}\cos(1-\epsilon)y - e^{-\epsilon x}\cos \epsilon y\right], \tag{2}$$

$$Y = \left(\frac{\epsilon}{1-\epsilon}\right)^{\epsilon}\frac{1-\epsilon}{\sin \epsilon\pi}\left[e^{(1-\epsilon)x}\sin(1-\epsilon)y + e^{-\epsilon x}\sin \epsilon y\right]. \tag{3}$$

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* *Berliner Sitzungsberichte*, April 23, 1868, p. 224 or *Wissenschaftliche Abhandlungen*, vol. 1, p. 154. A drawing is given by Maxwell, *Electricity and Magnetism*, vol. 1, fig. 12; and by Lamb, *Hydrodynamics*, p. 83.

If $y = \pi$, we have

$$X = -\left(\frac{\epsilon}{1-\epsilon}\right)^{\epsilon} \frac{1-\epsilon}{\tan \epsilon\pi} \left[e^{(1-\epsilon)x} + e^{-\epsilon x}\right], \tag{4}$$

$$Y = \left(\frac{\epsilon}{1-\epsilon}\right)^{\epsilon} (1-\epsilon) \left[e^{(1-\epsilon)x} + e^{-\epsilon x}\right], \tag{5}$$

and so

$$\frac{Y}{X} = -\tan \epsilon\pi; \tag{6}$$

that is, the line in the Z-plane corresponding to $y = \pi$ is a straight line making an angle of $-\epsilon\pi$ with the X-direction. Expressions (4) and (5) show that X is for this line always negative and Y positive. With $y = \pi$, X will be a maximum and Y a minimum when x satisfies the equation

$$e^{x} = \frac{\epsilon}{1-\epsilon}. \tag{7}$$

This substituted in (4) and (5) gives

$$X = -\frac{1}{\tan \epsilon\pi}, \quad Y = 1 \tag{8}$$

for the coordinates of the near end of the wall which extends thence to infinity toward the left and upward.

By referring back to (2) and (3) we see that if y were replaced by $-y$, X would remain as before while Y would have its sign changed. In particular, the line $y = -\pi$ will give a straight line in the Z-plane extending from the point $(-\cot \epsilon\pi, -1)$ to infinity toward the left and downward, making an angle of $\epsilon\pi$ with the X-direction. These two lines in the Z-plane are the traces of two plane walls which form a spout with an orifice 2 units in width.*

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* The function Z, regarded as a function of z, is single valued and analytic throughout the interior of the strip of the z-plane bounded by the lines $y = \pi$, $y = -\pi$; and this function maps the boundary of the strip on the lines of the Z-plane above considered. If we distinguish between the two sides of these lines, regarding the lines as doubly counting, then the boundary of the strip goes over in a one-to-one manner and continuously into the two edges of the slit made by cutting the Z-plane along the two lines in question. That the *interior* of the strip goes over conformally into the *whole Z-plane* exclusive of the cut, follows from an easy generalization of the theorem usually applied in problems of this sort. *Cf.* Picard, *Traité d' Analyse*, vol. 2, p. 280. [ED.]

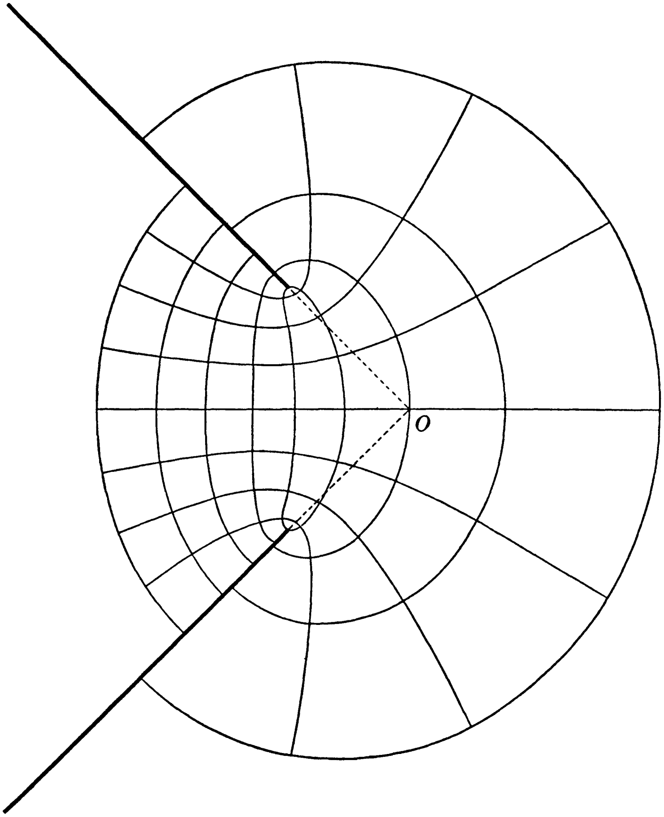
If we make $y = \beta$ and allow x to increase indefinitely, it follows from (2) and (3) that

$$\lim_{x=\infty} \frac{Y}{X} = \tan (1-\epsilon)\beta, \tag{9}$$

and similarly

$$\lim_{x=-\infty} \frac{Y}{X} = -\tan \epsilon\beta. \tag{10}$$

It thus appears that the stream line which corresponds to $y = \beta$ makes an angle of nearly $-\epsilon\beta$ with the X-direction when extended sufficiently far between the walls, and an angle of nearly $(1-\epsilon)\beta$ when extended sufficiently far outside of them.



Special Cases. If $\epsilon = \frac{1}{2}$, then (1), (2), and (3) become $Z = \sinh \frac{1}{2} z$, $X = \sinh \frac{1}{2} x \cos \frac{1}{2} y$, $Y = \cosh \frac{1}{2} x \sin \frac{1}{2} y$. $x = \alpha$ and $y = \beta$ transform into systems of confocal ellipses and hyperbolas whose foci are at the points $(0, \pm 1)$. The dihedral angle between the plane walls is 180°.

If $\epsilon = \frac{1}{4}$, we have the case shown in the accompanying figure. The walls make angles of ±45° with the X-direction, or an angle of 90° with each other. Equations (2) and (3) become

$$X = 0.806 \left[e^{\frac{3}{4} x} \cos \tfrac{3}{4} y - e^{-\frac{1}{4} x} \cos \tfrac{1}{4} y \right], \tag{12}$$

$$Y = 0.806 \left[e^{\frac{3}{4} x} \sin \tfrac{3}{4} y + e^{-\frac{1}{4} x} \sin \tfrac{1}{4} y \right]. \tag{13}$$

The values assigned to α range from $-\frac{3}{4}\pi$ to $\frac{1}{2}\pi$, and those assigned to β, from $-\pi$ to π. The equipotential line connecting the two foci, and setting out from them in the direction of the walls, is the line $e^{x} = \frac{\epsilon}{1 - \epsilon}$ when transformed to the Z-plane; it has not been drawn.

As ϵ becomes very small, the walls make a very small angle with the X-direction, and so, if produced, their intersection becomes removed too far to the right of the opening for a convenient X-origin. When ϵ approaches 0, the origin recedes indefinitely and we have in the limit the case given by Helmholtz. We can take the origin at the mouth of the spout if we introduce new variables z_1 and Z' such that

$$z_1 = z - \log \frac{\epsilon}{1 - \epsilon}, \qquad Z' = Z + \cot \epsilon\pi \tag{14}$$

If we now allow ϵ to approach 0 as its limit Z' approaches Helmholtz's function, Z_1, and we have in the limit:

$$Z_1 = \frac{1}{\pi} \left(e^{z_1} + z_1 \right) + \frac{1}{\pi}. \tag{15}$$

WASHINGTON, D. C., FEBRUARY, 1900.